AUGUST 2007 ANALYSIS QUALIFYING EXAM

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1. Problem 1

Note that $[0,1]$ is compact, so consider

$$\bigcup_{x\in[0,1]} I_x \supset [0,1]$$

By compactness we may extract a finite subcover $\{I_{x_1},\dots,I_{x_k}\}$. Let $\epsilon > 0$; for each $i = 1,\dots,k$, there exists $N_i \in \mathbb{N}$ such that for all $x \in I_{x_i}$,

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N_i$. Take $N := \max_i\{N_i\}$. For all $x \in [0,1]$, by construction we have that

$$|f_n(x) - f(x)| < \epsilon$$

whenever $n > N$¡ whence $f_n \to f$ uniformly.

2. Problem 2

(a). If f is integrable, this is trivial by Lebesgue's dominated convergence theorem. Assume then that f is not integrable. We may extract a subsequence f_{n_k} increasing to f, so that by Lebesgue's monotone convergence theorem,

$$\int f_{n_k} d\mu \to \int f = \infty$$

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Now, choose an arbitrary subsequence. We may extract a further subsequence that is increasing to f, and by the same logic above, this sub-subsequence must converge to $\int f$. Then, this shows that every subsequence has a further subsequence converging to f, which is equivalent to saying $\int f_n \to \int f$.

(b). If $\int f < \infty$, this is trivially true. Consider now $f_n := \chi_{[-n,n]}$. Then, $f_n \to 1$ everywhere, but,

$$\int 1 - f_n d\mu = \infty$$

for all n.

3. Problem 3

Define $f(x) := x^2$. Then, $f(A) = A^2$. Let $\delta > 0$. As $m(A) = 0$, we may find a sequence of open intervals $\{(a_k, b_k)\}$ with

$$A \subset \bigcup_{k=1}^{\infty}(a_k, b_k), \quad m\Big(\bigcup_{k=1}^{\infty}(a_k, b_k)\Big) < \epsilon$$

As f is continuous and surjective onto its image, every compact subset K' of $f(A)$ is the image of some compact subset K of A.

Then, by compactness we may select a finite subcover $\{(a_{k_1}, b_{k_1}), \dots, (a_{k_n}, b_{k_n})\}$ of K, so that

$$f(K) = K' \subset_{i=1}^{n} f(a_{k_i}, b_{k_i})$$

Let $\epsilon > 0$. Since f is absolutely continuous with bounded derivative on every compact subset, we may reselect δ such that

$$\sum_{i=1}^{n} b_{k_i} - a_{k_i} < \delta \implies \sum_{i=1}^{n} |f(b_{k_i}) - f(a_{k_i})| < \epsilon$$

As ϵ is arbitrary, we deduce that every compact subset of $m(f(A))$ has measure 0, whence

$$m(f(A)) = \sup_{K \subset f(A) \text{ cpt}} \{m(K)\} = 0$$

as contended.

4. Problem 4

(a). Let $\epsilon > 0$. Set $M_1 := \max_{x \in [a,b]} |f(x)|$, $M_2 := \max_{x \in [a,b]} |g(x)|$. We choose choose δ_1, δ_2 such that

$$\sum_{k=1}^{N} b_k - a_k < \delta_1 \implies \sum_{k=1}^{N} |f(b_k) - f(a_k)| < \frac{\epsilon}{2(M_2 + 1)}$$

$$\sum_{k=1}^{N} b_k - a_k < \delta_1 \implies \sum_{k=1}^{N} |g(b_k) - g(a_k)| < \frac{\epsilon}{2(M_1 + 1)}$$

Choose $\delta := \min\{\delta_1, \delta_2\}$. Then, whenever

$$\sum_{k=1}^{N} b_k - a_k < \delta$$

we have

$$\begin{aligned} \sum_{k=1}^{N} |f(b_k)g(b_k) - f(a_k)g(b_k)| &\leqslant \sum_{k=1}^{N} |f(b_k)g(b_k) - f(a_k)g(a_k)| \\ &\quad + \sum_{k=1}^{N} |f(a_k)g(b_k) - f(a_k)g(a_k)| \\ &\leqslant M_2 \cdot \sum_{k=1}^{N} |f(b_k) - f(a_k)| + M_1 \cdot \sum_{k=1}^{N} |g(b_k) - g(a_k)| \\ &< \frac{M_2 \epsilon}{2(M_2 + 1)} + \frac{M_1 \epsilon}{2(M_1 + 1)} \\ &< \epsilon \end{aligned}$$

So that $f \cdot g$ is absolutely continuous.

(b). Since $f \cdot g$ is absolutely continuous by part (a), we have

$$\begin{aligned} f(b)g(b) - f(a)g(a) &= \int_a^b (f \cdot g)'(x)dx \\ &= \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx \\ \implies \int_a^b f'(x)g(x)dx &= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx \end{aligned}$$

As desired.

5. Problem 5

Note that for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \lim_{n\to\infty} \int_1^\infty \frac{f_n(x)}{x} dx &\leqslant \lim_{n\to\infty} \Big(\int_1^\infty \frac{1}{x^q} dx \Big)^{1/q} \Big(\int_1^\infty |f_n|^p dx \Big)^{1/p} \quad \text{(Hölder's)} \\ &= \lim_{n\to\infty} \frac{1}{(1-q)^{1/q}} ||f_n||_p \\ &= \frac{1}{(1-q)^{1/q}} ||f||_p \end{aligned}$$

Since $f_n \to f$ and $f \in L^p(1,\infty)$, $||f_n||_p$ is a bounded sequence so that

$$f_n \leqslant \sup_n f_n \in L^p(1,\infty)$$

So that by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_1^\infty \frac{f_n(x)}{x} dx = \int_1^\infty \frac{f(x)}{x} dx$$

6. Problem 6

(a). We use Wirtinger derivatives for convenience. As f is holomorphic, $\frac{\partial f}{\partial \overline{z}} = 0$. This implies that $\frac{\partial f(\overline{z})}{\partial z} = 0$, in which case

$$\frac{\partial}{\partial \overline{z}} \overline{(f(\overline{z}))} = \overline{\frac{\partial f(\overline{z})}{\partial z}} = 0$$

So that $\overline{f(\overline{z})}$ is holomorphic.

(b). Note that $f(1/n) = \overline{f(\overline{1/n})}$, so that

$$g(z) := f(z) - \overline{f(\overline{z})}$$

is holomorphic by part (a). By the identity principle, since g is 0 on a set with an accumulation point, we deduce that $g(z) \equiv 0$. For $-1 < z < 1$, we have that $z \in \mathbb{R}$ so that

$$f(z) = \overline{f(z)}$$

Whence $f(z) \in \mathbb{R}$, as contended.

7. Problem 7

Note

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{e^x + e^{-x}} dx = \text{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{e^x + e^{-x}} dx$$

Consider the contour given in the problem statement, which we will denote by C. The only pole contained in this region is at $z = i\pi/2$. We also see that, by definition of C:

$$\begin{aligned}
\int_C \frac{e^{iz}}{e^z + e^{-z}} dz = & \int_0^{\pi} \frac{e^{iR-t}}{e^{R+it} + e^{-R-it}} dt \\
& + \int_0^{\pi} \frac{e^{-iR-t}}{e^{-R+it} + e^{R-it}} dt \\
& + \int_{-R}^{R} \frac{e^{ix}}{e^x + e^{-x}} dx \\
& + \int_{-R}^{R} \frac{e^{ix-\pi}}{e^{x+i\pi} + e^{-x-i\pi}} dx
\end{aligned}$$

Note first that

$$\begin{aligned}\operatorname{Res}\Big(\frac{e^{iz}}{e^z+e^{-z}}, i\pi/2\Big) &= \lim_{z\to i\pi/2}(z-i\pi/2)\frac{e^{iz}}{2\cosh(z)}\\ &= \frac{e^{-\pi/2}}{2\sinh(i\pi/2)}\\ &= \frac{-ie^{-\pi/2}}{2\sin(\pi/2)}\\ &= \frac{-ie^{-\pi/2}}{2}\end{aligned}$$

So that by Cauchy's residue theorem,

$$\int_C \frac{e^{iz}}{e^z+e^{-z}}dz = \pi e^{-\pi/2}$$

Letting $R\to\infty$,

$$\begin{aligned}\lim_{R\to\infty}\int_0^\pi \frac{|e^{iR+t}|}{|e^{-R+it}+e^{R-it}|}dz &\leqslant \lim_{R\to\infty}\frac{\pi e^\pi}{e^R-e^{-R}}\\ &= 0\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{R\to\infty}\int_0^\pi \frac{|e^{iR-t}|}{|e^{-R+it}+e^{R-it}|}dz &\leqslant \lim_{R\to\infty}\frac{\pi}{e^R-e^{-R}}\\ &= 0\end{aligned}$$

And,

$$\lim_{R\to\infty}\int_{-R}^R \frac{e^{ix}}{e^x+e^{-x}}dx = \int_{-\infty}^\infty \frac{e^{ix}}{e^x+e^{-x}}dx$$

So that combining all of the above, we are left with:

$$(1+e^{-\pi})\int_{-\infty}^\infty \frac{e^{ix}}{e^x+e^{-x}}dx = \pi e^{-\pi/2}$$

$$\implies \int_{-\infty}^\infty \frac{e^{ix}}{e^x+e^{-x}}dx = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}$$

And, taking the real part we are left with

$$\int_{-\infty}^\infty \frac{\cos(x)}{e^x+e^{-x}}dx = \frac{\pi}{2\cosh(\pi/2)}$$

8. Problem 8

We have uncountably many $a \in B_R(0)$, implying that for some $n \in \mathbb{N}$, $f^{(n)}(a) = 0$ for uncountably many $a \in B_R(0)$, since $\mathbb{N}$ is countable.

Consider now $f^{(n)}(z)$. If $f^{(n)}(z)$ is nonzero, then by holomorphicity its zeroes are isolated. But then there can only be countably many zeroes of $f^{(n)}$, since any set of isolated points in $\mathbb{C}$ is countable[1].

Thus, $f^{(n)}(z) = 0$ for all $z \in B_R(0)$, whence $f^{(k)}(z) = 0$ for all $k > n$, and we deduce that f is a polynomial of degree $n - 1$.

9. Problem 9

(a). True. We may write $f = g - h$ for g and h both monotone increasing functions. By monotonicity, the right limits of g and h exist, so the right limit of f exists.

(b). False. Let $f_n := n\chi_{[0,1/n]}$. Then, $f_n \to 0$ almost everywhere, but

$$\int f_n = 1 > 0 = \int 0$$

(c). False. The Cantor function is the counterexample, as it is continuous on a compact set, hence uniformly continuous. It is also a standard fact that it is of bounded variation. However, $f \neq \int f'$, in which case f is certainly not absolutely continuous.

(d). False. Hadamard's theorem gives that the radius convergence is $1/2$. However, being holomoprhic on the unit disk implies radius of convergence $\geqslant 1$.

[1]A quick way to see this is cover all isolated points by disjoint open neighborhoods. From each neighborhood, choose some rational coordinate. This induces an injection into $\mathbb{Q} \times \mathbb{Q}$, which is countable, so the set of isolated points is countable.

(e). True. This follows by the Casorati-Weierstrass theorem, which gives that for all $\epsilon > 0$, $g(B_\epsilon(0)\backslash\{0\})$ is dense in $\mathbb{C}$. By continuity and surjectivity of the exponential function, the image of $g(B_\epsilon(0)\backslash\{0\})$ remains dense in $\mathbb{C}$ for every $\epsilon > 0$. The only class of singularity with this property is an essential singularity, so that e^g must also have an essential singularity at 0.